Experimental validation of incipient failure of yield stress materials under gravitational loading

J. A. Chamberlain

Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

S. Clayton

Department of Chemical Engineering, University of Melbourne, Victoria 3010, Australia

K. A. Landman and J. E. Sader

Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

(Received 17 July 2002; final revision received 27 July 2003)

Synopsis

Recent theoretical work has investigated incipient failure of a cylinder of yield stress material under gravitational loading [Chamberlain et al., Int. J. Mech. Sci. 43, 793–815 (2001); 44, 1779–1800 (2002)]. This theoretical work, using the slip-line field method of plasticity, suggests that the height of incipient failure increases as the radius increases. In contrast, a simple heuristic model, the “uniform stress model,” predicts that the height of incipient failure is independent of the radius. We present detailed quantitative comparison of the slip-line and uniform stress models with experimental measurements, verifying the predictions of the slip-line model. © 2003 The Society of Rheology. [DOI: 10.1122/1.1619376]

I. INTRODUCTION

Concentrated suspensions such as mine tailings slurries or fresh concrete typically exhibit yield stress behavior. Under loading these suspensions behave as solids until a point is reached where the yield stress is exceeded, and the material begins to flow. This yielding behavior is important in the mining industry because of its impact on the pumping and disposal of mine wastes. It is also important in the construction industry, where knowledge of the consistency of fresh concrete is required. In contrast to “hard” materials such as metals and plastics, which have yield stresses in excess of 1 MPa, these concentrated suspensions typically exhibit low yield stress behavior, with yield stresses below 1 kPa. In these cases, failure induced by gravitational loading is commonly observed [Pashias et al. (1996)].

Recently, calculations of the incipient failure of a cylinder of yield stress material under gravitational loading have been performed using the slip-line field method of...
plasticity [Chamberlain et al. (2001); (2002)]. These calculations give the height of incipient failure, i.e., the height of the cylinder which causes the material to be on the verge of flowing due to gravitational loading. Results are expressed in scaled terms, with lengths scaled by \( \sigma_y / (\rho g) \) where \( \sigma_y \) is the uniaxial yield stress, \( \rho \) is the density, and \( g \) is the acceleration due to gravity. This slip-line field analysis will henceforth be referred to as the “slip-line model.”

The motivation for studying the incipient failure problem arises from the slump test method of measuring the yield stress of a slurry [Pashias et al. (1996)]. The slump test involves flow of a frustum of a cone or a cylinder of slurry whose initial height is greater than the height of incipient failure. Experimental data for this flow test have been commonly interpreted using the simple Murata model [Murata (1984); Christensen (1991); Pashias et al. (1996); Schowalter and Christensen (1998)] which assumes that only vertical stresses exist, and that these stresses are uniform across any horizontal plane. This model has been highly successful in predicting the amount of slump (reduction in height) occurring during the slump test. The assumptions inherent in this slump test model can also be used to predict the height of incipient failure. This implementation will henceforth be referred to as the “uniform stress model.”

Calculations show that the discrepancy between heights of incipient failure predicted by the uniform stress model and the slip-line model can be large, exceeding 100% for larger scaled radii [Chamberlain et al. (2002)]. We emphasize that incipient failure with a large scaled radius is outside the regime of parameters explored in existing slump test data, such as in Pashias et al. (1996), where the best agreement with the Murata model occurs for slump heights greater than 30%. Consequently, the discrepancy between the uniform stress and slip-line models is compatible with existing slump test data. As well, since the uniform stress model applies to incipient failure only, any conclusion drawn regarding the uniform stress model does not necessarily carry over as a similar conclusion for the original Murata model with a finite slump.

Here we investigate the validity of the uniform stress and slip-line models by carrying out experiments on materials with yield stresses below 1 kPa. Measurement of the height of incipient failure is performed by applying an external load to a cylinder of material whose height is less than the height of incipient failure (Fig. 1). Under certain conditions, discussed in Sec. III B, this external load simulates additional height, and thereby allows us to determine the height of incipient failure accurately. These experiments are used to assess the validity of the theoretical predictions of the slip-line model and the uniform stress model.
In the next section we will outline the main features and assumptions of the uniform stress and slip-line models. This will be followed by details of the experimental method in Sec. III. Analysis of the experimental results and general conclusions are then given in Secs. IV and V.

II. MODELS AND ASSUMPTIONS

We examine the effects of gravitational loading on a cylinder of yield stress material placed on a flat horizontal surface. This problem is illustrated in Fig. 1(a), where the radius and height of the cylinder are denoted by \( R_0 \) and \( H \), respectively. The magnitude of gravitational loading on the cylinder per unit volume is given by \( \rho g \). Also, because the cylinder is free standing, there is no traction on the vertical sides and horizontal top.

The material is assumed to be isotropic and homogeneous, with a constant uniaxial yield stress \( \sigma_y \). It is also assumed to be rigid-perfectly plastic in nature, i.e., rigid when stressed below the yield point, with no elastic effects.

The relationship between the uniaxial yield stress, \( \sigma_y \), and the shear yield stress, \( \tau_y \), depends on the type of yield condition describing the material. We consider two yield conditions, the Tresca condition with \( \sigma_y = 2\tau_y \) and the von Mises condition with \( \sigma_y = \sqrt{3}\tau_y \). The Tresca and von Mises yield conditions represent the extremes of behavior for the ratio of the yield stress in shear to the yield stress in uniaxial tension or compression. Specifying the yield condition allows measurements of the shear yield stress, performed with the vane rheometer, to be used in the slip-line and uniform stress models, which assume a uniaxial stress state. As well, it allows the shear yield stress measurements to be used in the analysis of incipient failure experiments, where the stress state may be very different from the simple shear state occurring with the vane rheometer. The Tresca and von Mises cases will be considered separately when interpreting the experimental data.

The height of incipient failure is defined to be the height of material \( H \) required to just initiate flow for a given radius \( R_0 \). We give theoretical predictions of this height based on balancing the gravitational loading, which tends to cause flow, with the strength of the cylinder and friction at the base, which resist flow. Two methods of calculating this height of incipient failure will be discussed: the first based on the uniform stress model and the second based on the slip-line model. In both models, the gravitational loading depends on the density, gravity, and the dimensions of the cylinder, while the strength of the cylinder is characterized by the uniaxial yield stress, \( \sigma_y \), and the radius of the cylinder, \( R_0 \). The two models differ in the way that the stresses are calculated within the cylinder. The uniform stress model (Sec. II A) makes heuristic assumptions about the stresses whereas the slip-line model (Sec. II B) calculates stresses rigorously and includes frictional forces on the base.

A. Uniform stress model

This model assumes that only vertical stress occurs in the cylinder and the stress is uniform across any horizontal plane. These assumptions imply that the magnitude of the vertical stress increases linearly with depth, from zero at the top to \( \sigma_y \) at a distance \( \sigma_y/(\rho g) \) below the top surface. Below this depth, there is a strip of plastic yielded material [Figs. 2(a) and 2(b)]. In this region the yield condition requires that the magnitude of the vertical stress is constant and equal to \( \sigma_y \), while the force balance equations require that the vertical stress increases with depth. These observations indicate that the strip of yielded material has infinitesimally small height [Fig. 2(c)]. Therefore, the uni-
form stress model predicts that the height of incipient failure is $\frac{\sigma_y}{\rho g}$, independent of the radius of the cylinder and the friction at the base. This prediction is tested against experimental data in Sec. IV.

B. Slip-line model

In contrast to the uniform stress model, the slip-line model rigorously calculates the stress field in the yielding region of the cylinder [Chamberlain et al. (2001); (2002)] using slip-line field theory. Both nonvertical and vertical stresses are taken into account. Also, friction between the material and the base is accounted for by assuming it is Coulombic in nature. The slip-line model predicts that there is a region of yielded plastic material of finite height at the base (Fig. 3), whereas the uniform stress model indicates that this yielded region is infinitesimally small [Fig. 2(c)].

The stress field calculated using the slip-line model can be used to predict the height of incipient failure, as shown in Fig. 4. As stated in the introduction, rather than using actual cylinder dimensions, these results are presented in terms of scaled quantities, with the height and radius of the cylinder scaled by the factor $\rho g/\sigma_y$. This allows the results to be applied to any yield stress material if the size of the cylinder is adjusted (see Sec. III B) according to the yield stress and density of the material.

The slip-line model predicts that the height of incipient failure increases with radius and is dependent on the coefficient of friction $\mu$ at the base, as illustrated in Fig. 4. The scaled height of incipient failure increases with greater friction at the base, until a critical value $\mu_c$ of the coefficient of friction is reached [Chamberlain et al. (2002)]. This critical value depends on the scaled radius of the cylinder, with larger scaled radii giving a larger critical coefficient of friction, up to a maximum critical coefficient of friction of $\mu_c = 2/(2 + \pi) \approx 0.39$ at $r_0 \approx 2$. Increasing friction above the critical value has no effect for friction coefficients above the critical value. The slip-line model predicts that the difference between the height of incipient failure predicted for the perfect slip ($\mu = 0$)

\begin{figure}[h]
\centering
\includegraphics[width=0.4	extwidth]{figure3.png}
\caption{Slip-line model prediction of the shape of the yielded region (shaded) for incipient failure. Three friction cases are shown: (a) perfect slip on the base ($\mu = 0$), (b) a perfectly rough base, and (c) intermediate friction ($\mu = 0.2$). In all cases the scaled radius is 1.0.}
\end{figure}
and perfectly rough ($\mu \geq \mu_c$) cases varies from zero at zero scaled radius, increasing to 18% at a scaled radius of 2.0. Also note that in Fig. 4 the case $\mu = 0.3$ is virtually indistinguishable from the case of a perfectly rough base. This dependence on radius and friction contrasts with the uniform stress model prediction (Fig. 4) that the height of incipient failure is independent of the radius and friction at the base. In the limit of infinitesimal scaled radii, $R_0 \rho g / \sigma_y \to 0$, the uniform stress and slip-line models give identical results, but for scaled radii greater than 4.0 differences between the two models exceed 75%.

III. MATERIALS AND EXPERIMENTS

A. Materials

The incipient failure experiments were performed on mine tailings slurries from the BHP Minerals Cannington silver/lead/zinc operation in Queensland, Australia [Clayton et al. (2003)].

Details of the samples used in the experiments are shown in Table I. The samples were characterized by measuring their shear yield stress using a Haake VT550 rheometer with vane attachment [Nguyen and Boger (1983); (1985)]. The vane was placed in the hand mixed sample and rotated slowly (0.2 rpm), recording the maximum torque $T_{\text{max}}$ and using the following equation from Nguyen and Boger (1985) to determine the yield stress.

FIG. 4. Predicted height of incipient failure, using the uniform stress and slip-line models, compared with experimental height of incipient failure data. The slip-line model results are shown for the friction cases $\mu = 0$ (slip), 0.05, 0.1, 0.2, 0.3, $\mu \geq \mu_c$ (stick). Two cases for the yield condition are shown: (a) Tresca and (b) von Mises. The error bars are calculated from the uncertainties in the raw data.
where $L$ and $D$ are the length and diameter of the vane, respectively. Here we have assumed that the wall and end shear stresses are equal to $\tau_y$ at the moment of yielding. At least three measurements were made with each sample, and the average and the standard error in the mean were calculated. The yield stress was found to be insensitive to the dimensions of the vane. As well, the density of each sample was measured, and the particle size distribution for the Cannington samples was typically by volume: $D_{20} = 8 \mu m$, $D_{50} = 32 \mu m$, $D_{80} = 100 \mu m$. The volume fraction of solids in the samples varied between 52% and 57%.

The behavior of the material for different mixing regimes was investigated by comparing the yield stress after hand mixing with mixing by a Heidolph RZR 2050 variable overhead mixer, fitted with a 40 mm diameter six bladed impeller. Four mixing regimes were used, (i) hand mixing, (ii) 30 min at 125 rpm, (iii) 30 min at 500 rpm, and (iv) 90 min at 500 rpm. Because the mechanical mixing generates heat, increasing evaporation, the results were compared by plotting a graph of yield stress versus solids concentration. There was no evidence of structural breakdown in these results. This contrasts with the results of Baudez et al. (2002) where thixotropy was observed in two mineral suspensions (a kaolin suspension and a cement paste). This illustrates how each mineral suspension exhibits unique behavior depending on the sample. Based on our investigation of different mixing regimes, we concluded that thixotropy was not significant in the experiments and hand mixing the Cannington material was an appropriate way to prepare the samples.

Slump tests have been performed on the Cannington samples by Clayton et al. (2003), with the material giving good agreement with the Murata slump test model for dimensionless slump heights greater than 0.25. We emphasize, however, that the incipient failure experiments cannot be compared directly with slump test results because the incipient failure experiments correspond to zero slump height and cover a range of scaled radii up to $\approx 2.0$.

When characterizing the samples, settling effects, homogeneity of the sample, and the experimental accuracy of the vane technique were considered. If a Cannington sample was left undisturbed for a few days, the slurry settled out with a layer of water above the more concentrated slurry below. The yield stress of this concentrated slurry was much higher than that of the original sample, typically greater than 1 kPa. Vigorous mixing was required to blend the high yield stress portion into the process water to form a more homogeneous suspension. In general, more mixing was required for samples with a higher volume fraction of solids, i.e., those with a higher yield stress in the final mixed state. Mixing also introduced air bubbles, which could not be completely removed in the

$$T_{\text{max}} = \frac{\pi D^3}{2} \left( \frac{L}{D + \frac{1}{3}} \right) \tau_y,$$  

(1)

### Table I. Details of Cannington samples used in experiments.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Shear yield stress [$\tau_y$ (Pa)]</th>
<th>Density [$\rho$ (kg m$^{-3}$)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>801±6.5</td>
<td>2120±9</td>
</tr>
<tr>
<td>B</td>
<td>891±28</td>
<td>2127±9</td>
</tr>
<tr>
<td>C</td>
<td>482±19</td>
<td>2195±9</td>
</tr>
<tr>
<td>D</td>
<td>846±16</td>
<td>2127±9</td>
</tr>
<tr>
<td>E</td>
<td>222±6</td>
<td>2098±9</td>
</tr>
</tbody>
</table>
time scale of the experiments. With air bubbles present as well as some local variation in
the volume fraction of the solid in liquid suspension, the samples were not perfectly
homogeneous. To estimate the errors in the yield stress due to inhomogeneity, at least
three vane rheometer measurements were taken for each sample. This gave a combined
estimate of the experimental error for the shear yield stress as well those due to inhomoge-
neity. The average and the standard error in the mean for these vane rheometer measure-
ments are shown in Table I. Each sample was mixed again immediately prior to perform-
ing the incipient failure experiments to reverse any settling that may have occurred after
the vane rheometer measurements were taken.

B. Incipient failure experiments

In each experiment an outer solid cylinder, the “supporting cylinder,” was filled to
create a cylinder of slurry. The dimensions of the cylinder of slurry need to satisfy two
conditions.

1. The height of the cylinder of slurry must be below the height of incipient failure;
otherwise the cylinder would slump with no applied force.
2. The plastic yielded region (Fig. 3) created during the experiment must remain below
the top plate (Fig. 5) when the cylinder of slurry yields.

The former condition ensures that the external load required to cause the cylinder to
yield can be easily measured. The latter condition is essential, otherwise the external
loading would not simulate additional height because it would disturb the internal struc-
ture of the yielded plastic region. As we shall discuss in Sec. III B (ii), the inside dimen-
sions of the supporting cylinder are not the same as the dimensions of the cylinder which
is tested for the height of incipient failure because the shape of the cylinder of slurry
changes during the experimental procedure. The earlier conditions apply to the dimen-
sions of the cylinder of slurry just prior to testing the height of incipient failure rather
than the inside dimensions of the supporting cylinder.

As a result of these conditions several supporting cylinders with carefully chosen
dimensions are required to cover a range of possible yield stresses. In general, higher
yield stress materials require larger cylinders and lower yield stress materials require
smaller cylinders. Table II gives a guide to appropriate dimensions of the cylinder of slurry (measured just prior to testing the height of incipient failure), in terms of the scaled height and radius, and the two extreme cases of friction at the base, perfect slip and a perfectly rough base. The minimum height is determined from condition (2) earlier using results in [Chamberlain et al. (2002)], and the maximum height is derived from the slip-line model curves in Fig. 4 and condition (1) earlier. Formulating this table in terms of scaled quantities shows how the cylinder needs to be matched with the yield stress and the density of the material. This requires an approximate knowledge of the yield stress prior to performing the experiments. We also note that there is a wider range of suitable scaled heights for cylinders with smaller scaled radii.

A series of combinations of materials and cylinders were chosen for the experiments (Table III). In each experiment, the supporting cylinder was placed on a horizontal perspex base and filled with the appropriate slurry sample. The main steps in the rest of the experimental procedure were as follows:

(i) measuring the coefficient of friction [Sec. III B (i)].

### Table II. Suitable cylinder dimensions for experiments.

<table>
<thead>
<tr>
<th>Scaled radius ((\rho g/\sigma_y)R_0)</th>
<th>Perfect slip</th>
<th>Perfectly rough base</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min.</td>
<td>Max.</td>
</tr>
<tr>
<td>0.50</td>
<td>0.37</td>
<td>1.18</td>
</tr>
<tr>
<td>1.00</td>
<td>0.64</td>
<td>1.32</td>
</tr>
<tr>
<td>1.50</td>
<td>0.85</td>
<td>1.44</td>
</tr>
<tr>
<td>2.00</td>
<td>1.03</td>
<td>1.54</td>
</tr>
<tr>
<td>2.50</td>
<td>1.18</td>
<td>1.64</td>
</tr>
<tr>
<td>3.00</td>
<td>1.32</td>
<td>1.72</td>
</tr>
</tbody>
</table>

### Table III. Experiments performed.

<table>
<thead>
<tr>
<th>Slurry sample (refer to Table I)</th>
<th>Supporting cylinder</th>
<th>Diameter (mm)</th>
<th>Height (mm)</th>
<th>Coefficient of friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>75</td>
<td>75</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>67</td>
<td>63</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>102</td>
<td>102</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>86</td>
<td>100</td>
<td>0.16</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>43</td>
<td>50</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>43</td>
<td>50</td>
<td>0.38</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>67</td>
<td>63</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>71</td>
<td>100</td>
<td>0.16</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>71</td>
<td>63</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>86</td>
<td>63</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>102</td>
<td>75</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>43</td>
<td>50</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>28</td>
<td>41</td>
<td>0.60</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>67</td>
<td>38</td>
<td>0.12</td>
<td></td>
</tr>
</tbody>
</table>
Sec. III B

These steps are explained later.

(i) Friction measurement

The coefficient of friction between the cylinder of slurry and the perspex base was measured by tipping up one end of the perspex base until the slurry and the supporting cylinder began to move. The angle of the base with the horizontal at the point of motion, $\phi$, gives an estimate of the coefficient of friction via the formula $\mu = \tan \phi$ where $\mu$ is the coefficient of friction. The measured values of the coefficient of friction are shown in Table III.

The presence of the supporting cylinder during measurement of the coefficient of friction introduces a small error in the coefficient of friction reported in Table III. Because the coefficient of friction between the supporting cylinder by itself and the base is approximately 0.55, the actual coefficient of friction between the base and the free standing cylinder of slurry is slightly less than the value in this table for all experiments except the experiment with a measured coefficient of friction of 0.60. The coefficient of friction values in Table III therefore slightly overestimate the actual coefficient of friction (except for the $\mu = 0.60$ experiment).

The variability in the coefficient of friction for sample D is possibly due to non-Coulombic friction, which is discussed in Sec. IV.

(ii) Lifting the supporting cylinder

The flat perspex base, with supporting cylinder containing slurry, was placed in a TA-XT2 texture analyzer. This instrument was used to accurately measure force and displacement over time as the cylinder was compressed.

Before the cylinder of slurry was compressed by the texture analyzer, the outer supporting cylinder was carefully lifted off. Interestingly, during this process the Cannington slurry increased in diameter as it came out of the supporting cylinder. Consequently, the material formed a free standing cylinder which was lower in height than the original supported cylinder. Some of this expansion is due to elastic effects, with yielding occurring in some cases as well. The changes in the geometry resulting from this deformation were taken into account in the analysis of the experiments by measuring the actual height and diameter of the sample just prior to the final compression.

The free standing cylinder exhibited yield stress behavior, i.e., the material did not flow within the time scale of experiments. Testing whether flow occurs over a longer time scale was not considered practical because the effect of drying and settling would dominate over any high viscosity creeping flow.

The top of the cylinder of slurry was not perfectly flat and horizontal when it came out of the supporting cylinder. Consequently, a small perspex plate was placed on top of the free standing cylinder (Fig. 5). An initial 10% compression was then applied by the platen to ensure the top plate and the top of the cylinder were horizontal (Fig. 5). The weight of the plate, $W_t$, which simulates an additional height of $\Delta H = W_t/(\pi \rho g R_t^2)$ is taken into account when the height of incipient failure is calculated.

When choosing the dimensions of the supporting cylinder, the changes in geometry during lifting the supporting cylinder and the 10% flattening of the top, as well as the adjustment for the weight of the top plate, need to be considered. The ranges of cylinder dimensions are as follows...
dimensions given in Table II need to be adjusted accordingly to give suitable ranges for the dimensions of supporting cylinders.

(iii) Final compression and determination of $H$

The diameter of the cylinder, $2R_0$, was measured before compressing the cylinder past failure to 50% of its height. During compression the force exerted by the platen and its vertical displacement were recorded (Fig. 6) at 0.2 s time intervals. The height of the cylinder before compression, $H_0$, was conveniently measured by the texture analyzer as the platen touched the top plate at the start of the 50% compression, taking into account the thickness of the top plate (Fig. 5). The inertial effects in the first 0.2 s as the top plate accelerates from rest to 1 mm/s can be estimated, and shown to be unimportant. The steep initial rise in force continues over typically 0.4 mm, followed by a gradual increase in force for the rest of the compression. The final state after compression is illustrated in Fig. 7.

Care was taken to identify the point where plastic flow commences on the force versus distance curve. This requires knowledge of the elastic properties of the material, which can be found by analyzing the texture analyzer data for the compression followed by an elastic rebound as the platen rises. These elastic properties were confirmed by a second

FIG. 6. Force vs displacement curve (solid) for incipient failure of sample D. This graph is from data obtained during the experiment illustrated in Figs. 5 and 7. The dotted line is a line of fit used to determine the yield point force as the intercept of this line with the vertical axis. In this case the yield point force is determined as 177 g weight before including the weight of the top plate.

FIG. 7. Cylinder of slurry (sample D from Fig. 5) after compression to 50% of the height in Fig. 5.
pass to the same depth. This gave an elastic strain limit of less than 3%, taking into account the elastic strain that is already present (due to gravity) before the texture analyzer applies a force. This result is consistent with yield point being the sharp transition at 0.4 mm in Fig. 6.

These force versus displacement data were found to be insensitive to the speed at which the platen was lowered, which was varied from 1 to 4 mm/s. This indicates that viscous forces are insignificant in these experiments, and the plasticity model used in the slip-line field calculations is appropriate for the Cannington material at platen speeds between 1 and 4 mm/s.

The force \( F \) required to cause the cylinder to yield was determined by taking a portion of the force versus displacement curve just after yielding, fitting a straight line, and extrapolating to the intercept with the vertical force axis, i.e., at zero displacement (Fig. 6). This technique works well if there is a steep initial rise in the force versus displacement data and a subsequent clear yield point when the force curve flattens out. The steep rise illustrated in Fig. 6 is more easily identified with cylinders whose height is below the maximum specified in Table II by a margin of 5% or more. The clarity of the yield point is improved by flattening the top of the cylinder with the platen and the top plate prior to final compression as discussed in Sec. III B (ii). In each experiment, the height of incipient failure \( H \) was calculated from the height \( H_0 \) and the yield point force \( F \) using

\[
H = H_0 + \frac{F + W_i}{\pi \rho g R_0^2}.
\]

IV. RESULTS AND DISCUSSION

All results for the height of incipient failure are presented using scaled quantities, with the radius of the cylinder and the height of incipient failure both scaled by the factor \( r_0/g \). As discussed in Sec. II, the yield condition determines how the uniaxial yield stress \( \sigma_y \) in this scaling factor relates to the shear yield stress \( \tau_y \) measured by the vane rheometer. We span a range of possible yield conditions by considering the Tresca and von Mises yield conditions [Lubliner (1990)] separately. The uniaxial yield stress for the Tresca material is greater by a factor of \( 2/\sqrt{3} \) when compared to the corresponding uniaxial yield stress for the von Mises material, assuming the same shear yield stress in both cases.

Experimental results for the scaled height of incipient failure versus the scaled radius are plotted in Fig. 4. Part (a) of this figure shows the Tresca analysis and part (b) shows the von Mises analysis. As the radius increases, the experimental values for the height of incipient failure show an increasing trend. The uniform stress model, represented by the horizontal lines in Fig. 4, does not agree with this trend in the experimental data. For small scaled radii and the Tresca case [Fig. 4(a)], the uniform stress model is roughly 15% below the experimental data, but for larger scaled radii the discrepancy increases, exceeding 100% at a scaled radius of 2. The discrepancy is even larger for the von Mises case [Fig. 4(b)].

The slip-line model, shown as a band over all base friction possibilities in Fig. 4, gives much better agreement with the experimental data. The slip-line model is within 14% of the experimental data for the Tresca case [Fig. 4(a)] and within 23% for the von Mises case [Fig. 4(b)] for all scaled radii. It also predicts the observed increasing trend in the height of incipient failure with scaled radius.

The underlying reason why the uniform stress model is more accurate for smaller scaled radii, and less accurate for larger radii, is that constraining the stress field to be
uniform across a horizontal plane is a better approximation for small scaled radii. With a small scaled radius, there is a small distance from the center to the free surface, giving only slight variations in the stress in a horizontal plane. For larger radii, there is more variation in the vertical stress and lateral stresses become important. In contrast to the uniform stress model, the slip-line model takes into account these nonuniform features of the stress field, giving the observed trend in the height of incipient failure versus the scaled radius.

While the slip-line model fits the data much better than the uniform stress model, there are several points (6 out of 14 for Tresca and 11 out of 14 for von Mises) above the band indicated for the range of possible friction conditions on the base. As discussed in [Chamberlain et al. (2002)], the slip-line solutions give a lower bound on the height of incipient failure rather than an exact solution. This lower bound property of the slip-line solutions is consistent with the experimental results.

While the lower bound property may explain the higher than expected height of incipient failure, other reasons for the higher than expected results were investigated. One of these reasons is that a fluid model with viscous effects and continuous velocity may be more appropriate than the plasticity model used in the slip-line fields. We expect that the boundaries of the deforming region (see Fig. 3), obtained with the slip-line model, correspond to regions of rapidly varying velocity in an equivalent fluid solution [see Davidson et al. (2000)]. If viscous effects were important, they would be particularly evident in these regions of potentially rapidly varying velocity. However, as discussed in Sec. II-B (iii), viscous forces are negligible in the experiments performed here and we conclude that a plasticity model is appropriate for the slow (1 mm/s) platen speeds used in the experiments.

Another potential source of error is in the variability of the friction measurements. However, the coefficient of friction is not used to calculate the height of incipient failure. This means that the observations of the height of incipient failure are valid, regardless of the friction measurement. In addition, even if the friction follows a non-Coulombic relationship, the two extremes of perfect slip and perfectly rough are still valid. The friction cannot be less than zero (perfect slip) and the friction force cannot violate the yield condition for material adjacent to the base (perfectly rough). This observation is reflected in the existence of a critical coefficient of friction (see Sec. II-B). It is not necessary to know the friction behavior accurately to compare the slip-line model perfectly rough and perfect slip curves (marked “stick” and “slip” in Fig. 4) to the experiments.

The friction condition on the base is not incorporated in the uniform stress model, which predicts that the height of incipient failure is independent of the friction condition on the base. The friction measurement therefore does not impact on comparison of the experimental height of incipient failure with the uniform stress model.

As discussed earlier, viscous forces, inertial effects, thixotropy, elastic effects, and variability in friction measurements do not explain the higher than expected results. Some other reasons for the higher than expected heights of incipient failure were considered. Inhomogeneity of the material and the imperfect shape of the cylinder could have caused random scatter but not the observed bias. According to the manufacturer’s specifications, the axial compliance of the texture analyzer gives a typical error in the displacement at the yield point of 0.02 mm, which is too small to affect the measurement of the yield point load as illustrated in Fig. 6. Pressure dependence in the yield stress, or some other deviation from the classical rigid-plastic behavior assumed by both the uniform stress and slip-line models, was a possibility. However, analysis of such effects is beyond the scope of the present study.
V. CONCLUSIONS

This experimental investigation leads to two conclusions.

1. The slip-line model with a Tresca yield condition agrees with the experimental height of incipient failure data to within 14%, and this model also predicts the correct increasing trend in height of incipient failure with scaled radius. This supports the validity of the slip-line model for predicting the height of incipient failure.

2. In contrast to the slip-line model, the uniform stress model underpredicts the experimental height of incipient failure. The discrepancy is greater than 100% for larger scaled radii and the prediction of no dependence on the scaled radius is incorrect. We therefore conclude that except for the case of small scaled radii, $R_0 \rho g/\gamma \ll 1$, the uniform stress model is inappropriate for predicting the height of incipient failure. This is in contrast to the slump test, which involves flow, where a model developed using similar assumptions can accurately predict the experimental behavior for dimensionless slump heights greater than or equal to 0.25.

ACKNOWLEDGMENTS

The authors would like to thank the Particulate Fluids Processing Center (PFPC), an Australian Research Council (ARC) Special Research Center, for financial support. The material used in the experiments was supplied by BHP Cannington. The authors would like to thank David Dunstan for access to the texture analyzer and Eugene Chai for assistance with the operation of the instrument. The authors would also like to thank David Boger, Barry Hughes, Christine Mangelsdorf, and Lee White for useful discussions.

References