2016 MIT Challenge

These are, primarily, algebraic solutions. For a numerical solution, see the spreadsheet 'MITSolutionsSpreadsheet.xlsx'

Question 1

Let:

a=amount spent on eradicating A

b=amount spent on eradicating B

$$Probabiliy\ of\ eradicating\ A = 1 - p_A e^{-L_A a}$$
$$Probabiliy\ of\ eradicating\ B = 1 - p_B e^{-L_B b}$$

Goal:

Maximise the sum of the probabilities of eradicating A & B (this is the same as maximising the number of species eradicated).

Maximise

$$H = (1 - p_A e^{-L_A a}) + (1 - p_B e^{-L_B b})$$

subject to the constraint

$$a + b = 10$$

To solve:

Substitute the constraint $b = 10 - a$ into H, so it becomes a function of a only.

$$H = (1 - p_A e^{-L_A a}) + \left(1 - p_B e^{-L_B (10-a)}\right)$$

To find a which maximises H, differentiate H with respect to a and set equal to 0:

$$H' = L_A p_A e^{-L_A a} - L_B p_B e^{-L_B (10-a)} = 0$$

Solving:

$$a = \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right) + 10L_B}{L_A + L_B}$$

$$b = 10 - a$$

Plugging in $L_A = 0.065, p_A = 0.75, L_B = 0.25, p_B = 0.8$, we get

$$a = 3.455199463$$
$$b = 6.544800537$$

So, the probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.400866023$$
$$1 - p_B e^{-L_B b} = 0.844225166$$

The expected number of species that will be eradicated is

$$H = 0.400866023 + 0.844225166 = 1.245091189$$

Alternative numerical solution (see spreadsheet)**:**
This can be done numerically by calculating H at $(a, b) = (a, 10 - a)$ combinations over values of a in the range 0 to 10, and finding the value of (a, b) that gives maximum H.

Solution:
The solution is to allocate about \$3.5M to eradicating species A, and \$6.5M to eradicating species B.
This will give about a 0.4 probability of eradicating species A and 0.84 probability of eradicating species B.
The expected number of species that will be eradicated is 1.25.

Question 2

New budget constraint:
This question is identical to question 1, except that now the budget constraint is $a + b = 8$, so the solution is:

$$a = \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right) + 8L_B}{L_A + L_B} = 1.867897876$$

$$b = 8 - a = 6.132102124$$

So, the probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.335749163$$
$$1 - p_B e^{-L_B b} = 0.827294782$$

The expected number of species that will be eradicated is:

$$H = 0.335749163 + 0.827294782 = 1.163043945$$

The percentage reduction in the probability of eradication compared with question 1 is:

$$Species\ A: \frac{0.400866023 - 0.335749163}{0.400866023} = 0.162440456$$

$$Species\ B: \frac{0.844225166 - 0.827294782}{0.844225166} = 0.020054346$$

Alternative numerical solution (see spreadsheet)**:**
This can be done numerically by calculating H at $(a, b) = (a, 8 - a)$ combinations over values of a in the range 0 to 8, and finding the value of (a, b) that gives maximum H.

Solution:
The solution is to allocate about \$1.9M to eradicating species A, and \$6.1M to eradicating species B.
This will give about a 0.34 probability of eradicating species A and 0.83 probability of eradicating species B.
The expected number of species that will be eradicated is 1.16.

Reducing the budget by $2,000,000 will reduce the probability of eradicating species A by about 16%. Spend it wisely!

Question 3

In this case, we think of U as the impact or damage cost that will be done by species B in some period, if it is not eradicated. Given that species A is twice as impactful as species B, the impact of A is *2U*.

Minimise:
Our goal is to minimise total expected impact:

$$I = 2Up_A e^{-L_A a} + Up_B e^{-L_B b}$$

subject to

$$a + b = 10$$

To solve:
Follow the same methodology as in questions 1 and 2:

$$I = 2Up_A e^{-L_A a} + Up_B e^{-L_B(10-a)}$$
$$I' = 2UL_A p_A e^{-L_A a} - UL_B p_B e^{-L_B(10-a)} = 0$$

Solving:

$$a = \frac{\ln\left(\frac{2L_A p_A}{L_B p_B}\right) + 10L_B}{L_A + L_B} = 5.655666703$$
$$b = 10 - a = 4.344333297$$

Note that U cancels out, so that only the relative effect is relevant.
The probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.480713959$$
$$1 - p_B e^{-L_B b} = 0.729971258$$

The expected number of species that will be eradicated is:

$$I = 0.480713959 + 0.729971258 = 1.210685217$$

The percentage reduction in the probability of eradication compared with question 1 is:

$$Species\ A: \frac{0.400866023 - 0.480713959}{0.400866023} = -0.199188585$$

$$Species\ B: \frac{0.844225166 - 0.729971258}{0.844225166} = 0.135335823$$

Alternative numerical solution (see spreadsheet)**:**
This can be done numerically by calculating I at $(a, b) = (a, 10 - a)$ combinations over values of a in the range 0 to 10, and finding the value of (a, b) that gives minimum I.

Solution:

The solution is to allocate about \$5.7M to eradicating species A, and \$4.3M to eradicating species B.
This will give about a 0.48 probability of eradicating species A (a 20% increase on Q1) and 0.73 probability of eradicating species B (about a 14% decrease).
The expected number of species that will be eradicated is now 1.21.

Question 4

This question could be interpreted in two ways:

Interpretation 1

Minimise:
Minimise the probability of failing to eradicate both species. Assuming that the two species are independent, this probability is:

$$J = p_A e^{-L_A a} p_B e^{-L_B b} = p_A p_B e^{-L_A a - L_B b}$$

We want to minimise this subject to

$$a + b = 10$$

To solve:
J is decreasing in both a and b. As $L_B > L_A$, allocate all funding to B in order to minimise J.

The probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.25$$
$$1 - p_B e^{-L_B b} = 0.934332001$$

Alternative numerical solution (see spreadsheet)**:**
This can be done numerically by calculating J at $(a, b) = (a, 10 - a)$ combinations over values of a in the range 0 to 10, and finding the value of (a, b) that gives minimum J.

The probability that both species won't be eradicated is:

$$J = p_A p_B e^{-L_A a - L_B b} = 0.049251$$

Solution:
The solution is to allocate all the budget to eradicating species B.
This will give about a 0.25 probability of eradicating species A and 0.93 probability of eradicating species B.
The probability that neither species is eradicated is 0.049.

Interpretation 2

Minimise:
Minimise the probability of failing to eradicate at least one species.
Assuming that the two species are independent, this probability is:

$$N = 1 - (1 - p_A e^{-L_A a})(1 - p_B e^{-L_B b})$$

We want to minimise this subject to

$$a + b = 10$$

To solve:

To find minimum, substitute the constraint $b = 10 - a$ into N, differentiate N with respect to a and set equal to 0:

$$N' = -L_A p_A e^{-L_A a}(1 - p_B e^{-L_B b}) + L_B p_B e^{-L_B(10-a)}(1 - p_A e^{-L_A a}) = 0$$

Simplifying:

$$L_A p_A e^{L_B(10-a)} - L_B p_B e^{L_A a} - (L_A - L_B) p_A p_B = 0$$

This must be solved numerically. Using solver in excel, the answer is

$$a = 5.07211305653351$$
$$b = 10 - a = 4.927886943$$

Alternative numerical solution (see spreadsheet)**:**
Alternatively, this answer could have been found numerically directly, by calculating N at $(a, b) = (a, 10 - a)$ combinations over values of a in the range 0 to 10, and finding the value of (a, b) that gives minimum N.

The probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.460638594$$
$$1 - p_B e^{-L_B b} = 0.766626543$$

The probability of failing to eradicate one or more species is

$$1 - (1 - p_A e^{-L_A a})(1 - p_B e^{-L_B b}) = 0.646862$$

Solution:
The solution is to allocate about $5.1M to eradicating species A, and $4.9M to eradicating species B.
This will give about a 0.46 probability of eradicating species A and a 0.77 probability of eradicating species B. The probability of failing to eradicate one or both species is 0.65.

Question 5

Note that:
$L_C = 0.3$
$p_C = 0.5$

Maximise:
Equation

$$G = (1 - p_A e^{-L_A a}) + (1 - p_B e^{-L_B b}) + (1 - p_C e^{-L_C c})$$

subject to budget

$$a + b + c = 10$$

To solve:
Set up the Lagrangian:

$$\Gamma = G + \lambda(10 - a - b - c)$$

Find the partial derivatives with respect to a, b, c and λ and set to 0:

$$\frac{\partial\Gamma}{\partial a} = L_A p_A e^{-L_A a} - \lambda = 0 \qquad (1)$$
$$\frac{\partial\Gamma}{\partial b} = L_B p_B e^{-L_B b} - \lambda = 0 \qquad (2)$$
$$\frac{\partial\Gamma}{\partial c} = L_C p_C e^{-L_C c} - \lambda = 0 \qquad (3)$$
$$\frac{\partial\Gamma}{\partial \lambda} = 10 - a - b - c = 0 \qquad (4)$$

So, we have four equations in four unknowns. To solve, first do (2)-(1), which becomes:
$L_A p_A e^{-L_A a} = L_B p_B e^{-L_B b}$

Rearranging gives:
$$a = \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right) + L_B b}{L_A}$$

Next, do (2)-(3) and substitute in (4), the budget constraint, for c (i.e. $c = 10 - a - b$)
$0 = L_B p_B e^{-L_B b} - L_C p_C e^{-L_C(10-a-b)}$

Rearranging gives:
$$b = \frac{\ln\left(\frac{L_B p_B}{L_C p_C}\right) + L_C(10 - a - b)}{L_B}$$

Sub in a
$$b = \frac{\ln\left(\frac{L_B p_B}{L_C p_C}\right) + L_C\left(10 - \left(\frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right) + L_B b}{L_A}\right) - b\right)}{L_B}$$
$$\left(L_B + \frac{L_C L_B}{L_A} + L_C\right) b = \ln\left(\frac{L_B p_B}{L_C p_C}\right) + L_C\left(10 - \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right)}{L_A}\right)$$
$$b = \frac{\ln\left(\frac{L_B p_B}{L_C p_C}\right) + L_C\left(10 - \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right)}{L_A}\right)}{L_B + \frac{L_C L_B}{L_A} + L_C} = 5.753345241$$

To get a, sub b into
$$a = \frac{\ln\left(\frac{L_A p_A}{L_B p_B}\right) + L_B b}{L_A} = 0.411140633$$

and to get c, sub a and b into
$c = 10 - a - b = 3.835514126$

The probabilities of eradication are:

$$1 - p_A e^{-L_A a} = 0.269777658$$
$$1 - p_B e^{-L_B b} = 0.810142191$$
$$1 - p_C e^{-L_C b} = 0.841785159$$

The expected number of species that will be eradicated is

$$H = 0.269777658 + 0.810142191 + 0.841785159 = 1.921705009$$

Alternative numerical solution (see spreadsheet)**:**
This can be done numerically by calculating G at $(a, b, c) = (a, b, 10 - a - b)$ combinations over values of a and b in the range 0 to 10, and finding the value of (a, b, c) that gives maximum G.

Solution:
The solution is to allocate about \$0.4M to eradicating species A, \$5.8M to eradicating species B, and \$3.8M to species C. This will give about a 0.27 probability of eradicating species A, a 0.81 probability of eradicating species B, and a 0.84 probability of eradicating species C.
The expected number of species that will be eradicated is 1.92.

Question 6

Each species could be included, or not included, so there are $2^3 = 8$ possible investment strategies. These are: null, A, B, C, AB, AC, BC, ABC, where AC for instance, means that we reduce the p for both A and C.

We find the optimal allocation of remaining funds under each strategy, and compare the expected number of species eradicated under each of the eight optimal allocations.

Analytically, calculating the allocation of remaining funds can be done as in 5, but adjusting the objective function (G) and constraint under each strategy.
the budget constraint changes for each strategy:

Strategy	G	Constraint
Null	$(1 - p_A e^{-L_A a}) + (1 - p_B e^{-L_B b}) + (1 - p_C e^{-L_C c})$	$a + b + c = 10$
A	$\left(1 - \frac{p_A}{2} e^{-L_A a}\right) + (1 - p_B e^{-L_B b}) + (1 - p_C e^{-L_C c})$	$a + b + c = 9$
B	$(1 - p_A e^{-L_A a}) + \left(1 - \frac{p_B}{2} e^{-L_B b}\right) + (1 - p_C e^{-L_C c})$	$a + b + c = 9$
C	$(1 - p_A e^{-L_A a}) + (1 - p_B e^{-L_B b}) + \left(1 - \frac{p_C}{2} e^{-L_C c}\right)$	$a + b + c = 9$
AB	$\left(1 - \frac{p_A}{2} e^{-L_A a}\right) + \left(1 - \frac{p_B}{2} e^{-L_B b}\right) + (1 - p_C e^{-L_C c})$	$a + b + c = 8$
AC	$\left(1 - \frac{p_A}{2} e^{-L_A a}\right) + (1 - p_B e^{-L_B b}) + \left(1 - \frac{p_C}{2} e^{-L_C c}\right)$	$a + b + c = 8$

BC	$(1 - p_A e^{-L_A a}) + \left(1 - \frac{p_B}{2} e^{-L_B b}\right) + \left(1 - \frac{p_C}{2} e^{-L_C c}\right)$	$a + b + c = 8$
ABC	$\left(1 - \frac{p_A}{2} e^{-L_A a}\right) + \left(1 - \frac{p_B}{2} e^{-L_B b}\right) + \left(1 - \frac{p_C}{2} e^{-L_C c}\right)$	$a + b + c = 7$

Note that the optimal allocation under the null case has already been found. Under the strategies A, AB, AC and ABC, following the steps in 5 results in a negative value of a. In these cases, we are at a border case, a=0, and the optimum allocation is found by allocating nothing to A and allocating between B & C using the method in question 1 but with an adjusted objective function to reflect the species and differing strategies under consideration.

This could also be done numerically, by comparing the expected number of species eradicated (G) under combinations of (a, b, c) for each strategy, given that the each strategy also has a differing budget constraint (see above table). We get the following maximum number of species eradicated under each strategy as:

Strategy	G
Null	1.921705010
A	2.24782464
B	2.002641338
C	1.982146672
AB	2.328812239
AC	2.309547864
BC	2.058546277
ABC	2.37728217

From this, we see that the maximum G across the strategies is under ABC, that is p is reduced for all species.

Alternatively, a simple heuristic shortcut is to compare the relative effects of spending an extra dollar on lowering p compared with increasing x on *f(x)*. Halving p has the effect of increasing *f(x)* by:

$$1 - \frac{p}{2} e^{-Lx} - (1 - pe^{-Lx}) = \frac{p}{2} e^{-Lx}$$

at each x, whereas increasing x has the effect of increasing *f(x)* by:

$$\frac{df}{dx} = Lpe^{-Lx}$$

at each x.

Note that $Lpe^{-Lx} < \frac{p}{2} e^{-Lx}$ as long as $L < 0.5$. So, halving p increases the eradication probability more than an extra investment in eradication at all levels of x, so long as $L < 0.5$.

As our goal is to maximise the sum of the three eradication probabilities, it follows that we should invest 1 million and decrease p for all species where

$L < 0.5$. In this case, $L_A = 0.065$, $L_B = 0.25$ and $L_C = 0.3$, so ABC is the optimal strategy.

Under this strategy, the optimal solution is to allocate nothing extra to A (except for the original 1M used to reduce p_A), so $a = 0$ and

$$c = \frac{\ln\left(\frac{L_C \frac{p_C}{2}}{L_B \frac{p_B}{2}}\right) + 10L_B}{L_C + L_B} = 2.658759868$$

$$b = 7 - c = 4.341240132$$

The individual probabilities of eradication are:

$$1 - \frac{p_A}{2} e^{-L_A a} = 0.625$$
$$1 - \frac{p_B}{2} e^{-L_B b} = 0.864881183$$
$$1 - \frac{p_C}{2} e^{-L_C b} = 0.887400986$$

Solution:
The solution is to reduce p for all three species, and then allocate the remaining \$7M as follows: Allocate nothing to eradicating species A, \$4.3M to eradicating species B, and \$2.7M to species C.
This will give a 0.625 probability of eradicating species A, about a 0.86 probability of eradicating species B, and a 0.89 probability of eradicating species C.
The expected number of species that will be eradicated is 2.38.