Upper and Lower Bounds for Incipient Failure in a Body Under Gravitational Loading

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Recent numerical work has investigated incipient failure of yield stress materials under gravitational loading, for both the rectangular block and cylinder geometries [Chamberlain et al.: 2001, Int. J. Mech. Sci. 43(3):793-815, 2002, Int. J. Mech. Sci. 44(8):1779-1800]. While the rectangular block solution is exact, the cylinder solutions give lower bounds on the height of incipient failure. Consequently, we construct upper bound solutions for the height of incipient failure of a cylinder under gravitational loading. This closes the cylinder problem and quantifies the accuracy of the Haar-Karman hypothesis used in slip-line analysis. For completeness, we also give a simple lower bound solution for the cylinder, as well as upper and lower bound solutions for the two-dimensional rectangular block. These results have the advantage of being analytical, in contrast to the previous purely numerical results. [DOI: 10.1115/1.1767164]

1 Introduction

Flow of a cylinder of yield stress material under gravity to a lower height can be used to determine its yield stress; the relevant experimental technique is commonly termed the “slump test” (Murata [1], Christensen [2], and Pashias et al. [3]). While this flow phenomenon has been studied extensively, the closely related problem of incipient failure, where the body is on the verge of flowing, has received limited attention in the literature, with the only work to date being the slip-line analyses of Chamberlain et al. [4,5]. While an exact solution was given for the case of a rectangular block (Chamberlain et al. [5]), only a lower bound solution was derived for the complementary case of a cylinder (Chamberlain et al. [4]) with the Haar-Karman hypothesis being invoked. The principal aim of this article is therefore to close the cylinder problem by using limit analysis to construct an upper bound on the height of incipient failure. For completeness, we also present upper and lower bound solutions for the rectangular block. The analytical formulas obtained using upper and lower bound analyses are of practical value due to their simplicity.

The geometry and coordinate systems used are shown in Fig. 1. Final results are scaled by the length $2r_s/(\rho g)$, where $r_s$ is the shear yield stress, $\rho$ is the density and $g$ is the acceleration due to gravity, giving the scaled radius or half-width $r_0$ and the scaled height of incipient failure $h$. In the analyses we use both Tresca and von Mises yield conditions (Desai and Siriwardane [6]).

2 Lower Bound Analyses

To construct a lower bound solution, we specify a statically admissible stress field.

Plane Strain Rectangular Block. An appropriate statically admissible stress field for the rectangular block is

$$\Sigma = -\rho g \left( \frac{2\tau_s}{\rho g} \right) \hat{X}_2 \hat{X}_2 + \left( \frac{\tau_s}{\rho g} \right) \hat{X}_3 \hat{X}_3,$$

where the height of the block is

$$H_b = \frac{2\tau_s}{\rho g}.$$

Using the lower bound theorem of limit analysis (Chakrabarty [7]) it then follows that the height of incipient failure is greater than or equal to $H_b$, i.e., in scaled terms

$$h \geq 1.$$

Axisymmetric Cylinder. The statically admissible stress field for the cylinder case is chosen as

$$\Sigma = -\rho g \left( \frac{\sigma_y}{\rho g} - Z \right) \hat{Z} \hat{Z},$$

with the height of the body, $H_b$, equal to $\sigma_y/(\rho g)$, where $\sigma_y$ is the uniaxial yield stress. In scaled terms, the lower bound results are

$$h \geq 1 \quad \text{(Tresca)} \quad \text{and} \quad h \geq \frac{\sqrt{3}}{2} \quad \text{(von Mises)}.$$

3 Upper Bound Analyses

To use the upper bound theorem of limit analysis we require a kinematically admissible velocity field (Chakrabarty [7]). We

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consider two extremes of base friction: (i) perfect slip and (ii) a perfectly rough base.

**Plane-Strain Rectangular Block**

**Perfect Slip Base.** The flow field is approximated by rigid regions separated by velocity discontinuities, as illustrated in Fig. 2(a). The kinematically admissible velocity field is

\[ v^* = \begin{cases} 
-\hat{X}_2 & \text{above } AB, \\
\cot \zeta \hat{X}_1 & \text{below } AB, \\
0 & \text{below } AC,
\end{cases} \quad (6) \]

from which we obtain (Chakrabarty [7]) the upper bound

\[ h \leq \sqrt{1 + r_0}. \quad (7) \]

**Perfectly Rough Base.** To satisfy the perfectly rough condition on the base, we require two discontinuities, see Fig. 2(b). The kinematically admissible velocity field is given by

\[ v^* = \begin{cases} 
-\hat{X}_2 & \text{above } AB, \\
\mu_1 \hat{X}_1 + \mu_2 \hat{X}_2 & \text{between } AB \text{ and } AC, \\
0 & \text{below } AC,
\end{cases} \quad (8) \]

where

\[ \mu_1 = \frac{1}{\tan \zeta + \tan \gamma}, \quad \mu_2 = \frac{-\tan \gamma}{\tan \zeta + \tan \gamma}. \quad (9) \]

The upper bound result is then

\[ h \leq \sqrt{1 + 2r_0}. \quad (10) \]

**Axisymmetric Cylinder.** For the axisymmetric geometry, we use configurations of velocity discontinuities identical to those used in the plane strain solutions (see Fig. 2). In addition, we specify that the radial velocity is constant (Kudo [8]).

**Perfect Slip Base.** Referring to Fig. 2(a), the velocity field is specified as

\[ v^* = \begin{cases} 
-\hat{Z} & \text{above } AB, \\
\frac{1}{2b} Z \hat{Z} & \text{below } AB,
\end{cases} \quad (11) \]

where \( b = \tan \zeta \), and \( AB \) is a velocity discontinuity.

**Tresca Yield Condition.** Applying the upper bound inequality and minimizing with respect to \( b \), we obtain

\[ 4h \leq \frac{1}{b} + b + 2r_0b + \frac{1}{2} \sqrt{1 + b^2} + \frac{1}{2b} \ln(\sqrt{1 + b^2} + b). \quad (12) \]

where \( b \) satisfies

\[ 2b^2(1 + 2r_0) + b \sqrt{1 + b^2} = 2 + \ln(b + \sqrt{1 + b^2}). \quad (13) \]

**von Mises Yield Condition.** Applying the upper bound inequality and minimizing with respect to \( b \), we obtain

\[ 4h \leq \frac{1}{b} + b + 2r_0b + \sqrt{1 + \frac{b^2}{4} + \frac{2}{b} \ln\left(\sqrt{1 + \frac{b^2}{4}} + \frac{b}{2}\right)}. \quad (14) \]

where

\[ b^2(1 + 2r_0) + b \sqrt{1 + \frac{b^2}{4}} = 1 + 2\ln\left(\sqrt{1 + \frac{b^2}{4}} + \frac{b}{2}\right). \quad (15) \]
Perfectly Rough Base. Referring to Fig. 2(b), the velocity field \( \mathbf{v}^* \) is given by

\[
\mathbf{v}^* = \begin{cases} 
-\mathbf{Z} & \text{above } AB, \\
\frac{1}{2t} \left[ \mathbf{R} + \frac{1}{R} \left[ \alpha(t(R_0 - 2R) - Z) \right] \right] & \text{between } AB \text{ and } AC, \\
0 & \text{below } AC,
\end{cases}
\]

where \( \alpha \) and \( t \) are defined by \( \alpha t = \tan \gamma \) and \( (1 - \alpha)t = \tan \xi \).

Tresca Yield Condition. Applying the upper bound inequality, minimizing with respect to \( a \) and \( t \) gives the required result

\[
4h \leq \frac{2}{t} + \frac{t}{2} + 2\mu t + \frac{1}{2} \sqrt{1 + \frac{t^2}{4} + \frac{1}{2t} \ln \left( \frac{\sqrt{4 + t^2} + t}{\sqrt{4 + t^2} - t} \right)},
\]

where

\[
4 + \ln \left( \frac{\sqrt{4 + t^2} + t}{\sqrt{4 + t^2} - t} \right) = (1 + 4r_0)t^2 + \frac{t}{2} \sqrt{4 + t^2}.
\]

von Mises Yield Condition. Applying the upper bound inequality and minimizing with respect to \( a \) and \( t \) gives

\[
4h \leq \frac{2}{t} + \frac{t}{2} + 2\mu t + \sqrt{1 + \frac{t^2}{16} + \frac{2}{t} \ln \left( \frac{\sqrt{16 + t^2} + t}{\sqrt{16 + t^2} - t} \right)}.
\]

4 Results and Discussion

The above results are illustrated in Figs. 3, 4, and 5. For the rectangular block we find that the upper bound solutions give an excellent approximation to the exact slip-line results, whereas the lower bound solutions are comparatively poor.

For the cylinder case, the actual solution must lie between the slip-line lower bound and the appropriate upper bound solution, possibly touching one of these curves (see Figs. 4 and 5). Consequently, the error in the height of incipient failure introduced by using the Haar-Karman hypothesis is bounded by the difference between the slip-line lower bound solution and the upper bound solution. This difference depends on the yield condition, base boundary condition, and radius, and is quantified in Figs. 3 and 4.

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References


